

WRDC-TR-89-3097

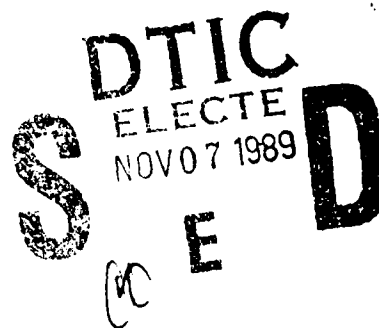
# AN ALTERNATIVE DERIVATION FOR AN INTEGRAL EQUATION FOR LINEARIZED SUBSONIC FLOW OVER A WING



Marc H. Williams, Karl G. Guderley, and Mark R. Lee

Aeroelasticity Group  
Structures Division

August 1989



Final Report for the Period February 1988 to January 1989

Approved for public release; distribution unlimited.

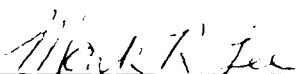
FLIGHT DYNAMICS LABORATORY  
WRIGHT RESEARCH AND DEVELOPMENT CENTER  
AIR FORCE SYSTEMS COMMAND  
WRIGHT-PATTERSON AIR FORCE BASE, OHIO 45433-6553

89 11 06 080

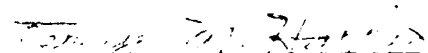
## NOTICE

When government drawings, specifications, or other data are used for any purpose other than in connection with a definitely Government-related procurement, the United States Government incurs no responsibility or any other obligation whatsoever. The fact that the Government may have formulated or in anyway supplied the said drawings, specifications, or other data is not to be regarded by implication, or otherwise in any manner, as licensing the holder or any other person or corporation; or as conveying any rights or permission to manufacture, use, or sell any patented invention that may in any way be related thereto.

This technical report has been reviewed and is approved for publication.



MARK R. LEE  
Project Engineer  
Aeroelasticity Group



TERRY M. HARRIS, Technical Manager  
Aeroelasticity Group  
Analysis & Optimization Branch

## FOR THE COMMANDER



JOHN T. ACH, Chief  
Analysis & Optimization Branch  
Structures Division

If your address has changed, if you wish to be removed from our mailing list, or if the addressee is no longer employed by your organization please notify WRDC FIBRC, WPAFB, OH 45433-6553 to help us maintain a current mailing list.

Copies of this report should not be returned unless return is required by security considerations, contractual obligations, or notice on a specific document.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE

## REPORT DOCUMENTATION PAGE

Form Approved  
OMB No. 0704-0188

1a REPORT SECURITY CLASSIFICATION UNCLASSIFIED			1b RESTRICTIVE MARKINGS		
2a. SECURITY CLASSIFICATION AUTHORITY			3 DISTRIBUTION/AVAILABILITY OF REPORT Approval for public release; distribution is unlimited.		
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE			4 PERFORMING ORGANIZATION REPORT NUMBER(S) WRDC-TR-89-3097		
5 MONITORING ORGANIZATION REPORT NUMBER(S)			6a. NAME OF PERFORMING ORGANIZATION Flight Dynamics Laboratory Wright Research & Develop Center		
6b. OFFICE SYMBOL (If applicable) WRDC/FIBRC			7a. NAME OF MONITORING ORGANIZATION		
6c. ADDRESS (City, State, and ZIP Code) Wright-Patterson AFB OH 45433-6553			7b. ADDRESS (City, State, and ZIP Code)		
8a. NAME OF FUNDING/SPONSORING ORGANIZATION			8b. OFFICE SYMBOL (If applicable)		
9 PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER			8c. ADDRESS (City, State, and ZIP Code)		
10 SOURCE OF FUNDING NUMBERS			11 TITLE (Include Security Classification) An Alternative Derivation for an Integral Equation for Linearized Subsonic Flow Over a Wing		
PROGRAM ELEMENT NO 62201F	PROJECT NO 2401	TASK NO 02	WORK UNIT ACCESSION NO. 73		
12 PERSONAL AUTHOR(S) Williams, Marc H., Guderley, Karl G., and Lee, Mark R.					
13a. TYPE OF REPORT Final		13b. TIME COVERED FROM Feb 88 TO Jan 89		14 DATE OF REPORT (Year, Month, Day) August 1989	
15. PAGE COUNT 41					
16 SUPPLEMENTARY NOTATION					
17. COSATI CODES			18 SUBJECT TERMS (Continue on reverse if necessary and identify by block number)		
FIELD	GROUP	SUB-GROUP	linearized subsonic time dependent flow, unsteady aerodynamics; integral equations. <i>C.D. 176</i>		
01	01				
20	04				
19. ABSTRACT (Continue on reverse if necessary and identify by block number) An integral equation for the time dependent linearized subsonic flow over a wing has been derived in a previous report by K. G. Guderley and Maxwell Blair. In this report, written by Williams, Guderley, and Lee, an alternative derivation due to Marc H. Williams is presented. The difference lies in the sequence of events. The original derivation was carried out in a coordinate system moving with the wing. Williams' derivation makes the crucial step in a coordinate fixed in the undisturbed air and then carries out a transformation to wing coordinates. In addition, this derivation clarifies the relation between the integral equation for steady and unsteady flow, and derives a formulation based on the Lorentz transform.					
20 DISTRIBUTION/AVAILABILITY OF ABSTRACT <input checked="" type="checkbox"/> UNCLASSIFIED/UNLIMITED <input type="checkbox"/> SAME AS RPT <input type="checkbox"/> DTIC USERS			21 ABSTRACT SECURITY CLASSIFICATION UNCLASSIFIED		
22a NAME OF RESPONSIBLE INDIVIDUAL Mark R. Lee			22b TELEPHONE (Include Area Code) (513) 255-7384		22c OFFICE SYMBOL WRDC/FIBRC

## FOREWORD

This Technical Report was prepared by the Aeroelasticity Group of the Analysis and Optimization Branch, Structures Division, Flight Dynamics Laboratory, Wright Research and Development Center, Wright-Patterson Air Force Base, Ohio. Under a previous contract, the senior author treated the time dependent integral equation for the linearized potential flow over a wing (Reference 1). The present report shows an alternative and somewhat simpler derivation due to Professor Marc H. Williams (Purdue University). The report constitutes a somewhat extended version of Professor Williams' work. The contribution of the senior authors was made without contractual obligation in order to round out the previous investigation of Reference 1. Part of the present work was performed by the junior author in the Aeroelasticity Group under Project 2401 "Structural Mechanics." Task 240102 "Design and Analysis Methods for Aerospace Vehicle Structures." Work Unit 24010273 "Aeroservoelasticity." This manuscript was released in August 1989 for publication as a Technical Report. This report covers work conducted from February 1988 to January 1989.

<b>Accession For</b>	
NTIS GRA&I	<input checked="" type="checkbox"/>
DTIC TAB	<input type="checkbox"/>
Unannounced	<input type="checkbox"/>
Justification	
By _____	
Distribution/	
<b>Availability Codes</b>	
Dist	Avail and/or Special
<b>A-1</b>	

## TABLE OF CONTENTS

Section	Page
I      Introduction	1
II     Basic Equations in Acoustic Coordinates	2
III    Wing Coordinates	12
IV    Lorentz Coordinates	19
V     Identity of Equations (3.21) and Equation (VI.18) of Reference 1	23
VI    Steady State Equations	25
VII   References	33
Appendix: Identity of the Present Equations (3.17), (3.18), and (3.22) with Reference 1	34

## Section I

### Introduction

For the linearized acceleration potential, as well as the velocity potential equation in three-dimensional unsteady flows, there exists a fundamental solution in a closed form. This makes it possible to formulate the problem of the flow over a wing (or also for the whole airplane) in terms of an integral equation, by which the problem is reduced from three space dimensions and time to two space dimensions and time. Under special conditions, for instance the flow over an oscillating wing, the problem has been solved before; an example is the doublet-lattice method. In these approaches, valid for steady or oscillatory flows, the integral equation formulation does not appear explicitly. It is by-passed by an immediate discretization of the problem combined with a specialization of the fundamental solution. By this procedure one avoids the need to deal with the singularities which by necessity occur in the integral equation.

The authors of Reference 1 (Guderley and Blair) believe that for general applications which deal with the problem in the time domain, the explicit integral equation may be advantageous. However, to bring the integral equation into a form suitable for a numerical approach it is necessary to carry out transformations which weaken certain singularities otherwise encountered. The analytical work carried out for this purpose in Reference 1 is rather formidable. By performing the necessary steps in a different order, Professor Marc H. Williams has accomplished considerable simplifications. Moreover, Professor Williams' approach leads to a simplification of one of the final equations and makes a complete exploration of the relations between the steady and the unsteady problem possible. In addition, it opens the way to a rather intriguing development, namely the treatment of the integral equation by means of Lorentz coordinates.

The present report, prepared by K.G. Guderley and Mark R. Lee, follows the notes kindly sent to Guderley by Professor Williams. Some of the intermediate steps have been filled in, to facilitate the study of the report for a reader, who on one hand does not want to accept the results on faith but on the other hand has only limited experience in, or time for, the necessary mathematical manipulations.

## Section II

### Basic Equations in Acoustic Coordinates

Let  $x^0, y^0, z^0$  be a Cartesian system of coordinates which is at rest with respect to the undisturbed air. The superscript zero is applied because at some later stage the coordinates  $x, y, z$  will be introduced for a Cartesian system fixed with respect to the wing. With respect to this (no superscript) system, the undisturbed air moves from left to right with constant speed in the  $x$ -direction. The planform of the wing lies in the  $x, y$ -plane. Let  $i, j, k$  be unit vectors in the  $x^0, y^0, z^0$  or  $x, y, z$ -directions respectively. Let

$$\vec{r}^0 = x^0 i + y^0 j + z^0 k \quad (2.1)$$

Accordingly  $\vec{r}^0$  represents a vector with components  $x^0, y^0, z^0$ . The reader should not confuse the vector  $\vec{r}^0$  with its component in the  $i$  direction, denoted by  $x^0$ . In the  $x^0, y^0, z^0$ -system the wing moves with constant velocity in the direction of decreasing  $x^0$ . In this process it sweeps out a strip in the  $x^0, y^0$  plane which has a width (i.e. an extension in the  $y^0$ -direction) equal to the span. In the  $x^0$ -direction this strip extends from the position of the wing at the time when the unsteady motion started to its position at the current time. In the present analysis viscosity is disregarded and the air is considered as isentropic. Since the initial state of the air is free of vorticity, the Helmholtz vorticity laws are applicable, and the velocities can be described as the gradient of a potential (denoted by  $\phi$ ). The equation for the potential is obtained by the requirement that the equation of conservation of mass be satisfied. We make the further assumption that the perturbation generated in the air (which was originally at rest) is small. Then the equation for the potential reads

$$a_0^2 \nabla^{02} \phi^0 = \frac{\partial^2 \phi^0}{\partial t^2} \quad (2.2)$$

Here  $a_0$  is the velocity of sound in the unperturbed state, and  $\nabla^{02}$  is the Laplace operator in the  $x^0, y^0, z^0$ -system. The del operator  $\nabla^0$  is defined by

$$\nabla^0 \equiv i \frac{\partial}{\partial x^0} + j \frac{\partial}{\partial y^0} + k \frac{\partial}{\partial z^0} \quad (2.3)$$

Let  $\vec{\xi}$  be a fixed vector in the  $x^0, y^0, z^0$ -space.

$$\vec{\xi} = i \xi + j \eta + k \zeta \quad (2.4)$$

and

$$\vec{R} = R = \left[ (x^0 - \xi)^2 + (y^0 - \eta)^2 + (z^0 - \zeta)^2 \right]^{1/2} \quad (2.6)$$

Then a fundamental solution with singularity at the point  $\vec{x} = \vec{\xi}$  is given by

$$\phi^s(\vec{x}, \vec{\xi}, t) = -\frac{1}{4\pi} \frac{1}{R} h(\tau) \quad (2.7)$$

where

$$\tau = t - \left( \frac{R}{a} \right) \quad (2.8)$$

$R/a$  is the time it takes a perturbation to travel from the point  $\vec{\xi}$  to  $\vec{x}$ . Obviously the solution (2.7) has spherical symmetry (the center of the sphere is given by  $\vec{x} = \vec{\xi}$ ). The mass flow per unit of time through a sphere with radius  $R_c$  is given by

$$-4\pi R_c^2 \frac{1}{4\pi} \left[ -\frac{h(\tau)}{R_c^2} - \frac{1}{a} \frac{1}{R_c} h'(\tau) \right]$$

where the prime denotes the differentiation with respect to the argument. For  $R_c \rightarrow 0$ , one has  $\tau \rightarrow t$ , and the last expression reduces to  $h(t)$ . Accordingly,  $h(t)$  is the mass flow per second. This is the interpretation of the function  $h(t)$ . While this discussion has no bearing on the following development, it does explain the factor  $\left( \frac{-1}{4\pi} \right)$ .

At the point given by  $\vec{x} = \vec{\xi}$  the potential equation is not satisfied, for there is a mass flow emerging from this point. The flow will be represented by a superposition of particular solutions, Eq.(2.7), with centers at different points  $\vec{\xi}^o$ . The functions  $h^o$  are allowed to be different for each value of  $\vec{\xi}^o$ . This is expressed by writing  $h^o(\vec{\xi}^o, \tau)$ . Notice that  $\vec{\xi}^o$  also occurs in the definition of  $\tau$ , thus  $\vec{\xi}^o$  appears in  $h^o$  twice, implicitly (in  $\tau$ ) and explicitly as first argument. The singular points  $\vec{x} = \vec{\xi}^o$  cannot lie inside of the flow field; and they can appear only within the strip of the  $x^o, y^o$ -plane swept out by the wing. The component  $\xi^o_z$  of the vector  $\vec{\xi}^o$  is therefore zero. This assumes the wake to be flat.

The unsteady up and down motion of the wing generates at its surface a velocity in the  $z'$ -direction which is the same on the upper and lower side of the wing. The field of the  $z'$  components of the velocity is therefore symmetric with respect to the  $x^o, y^o$ -plane. The potential is then anti-symmetric with respect to this plane. This implies for points in the  $x', y'$ -plane outside of the strip swept out by the wing that the potential is zero. The source potential of Eq. (2.7) is spherically symmetric; therefore, it is symmetric with respect to the  $x, y$ -plane if  $\vec{\xi} = i\xi^o + j\eta^o$ . An antisymmetric expression is obtained by a differentiation with respect to  $z'$ . The resulting expression can be interpreted as a doublet (of moment -1) oriented in the positive  $\vec{z}$  direction (actually this interpretation



is unimportant). Thus we represent the potential by a superposition of "doublets"

$$\phi^d(\vec{x}^\circ, \vec{\xi}^\circ, t) = \frac{-1}{4\pi} \frac{\partial}{\partial z^\circ} \frac{h^\circ(\vec{\xi}^\circ, \tau)}{R_\circ}$$

where  $\vec{\xi}^\circ = i\xi^\circ + j\eta^\circ$ , is restricted to the above mentioned strip of the  $x^\circ, y^\circ$ -plane for the remainder of the report. Accordingly, the potential will appear in the form

$$\phi^\circ(\vec{x}^\circ, t) = -\frac{1}{4\pi} \frac{\partial}{\partial z^\circ} \int \int_A \frac{1}{R_\circ} h^\circ(\vec{\xi}^\circ, \tau) d\xi^\circ d\eta^\circ \quad (2.9)$$

The unbrl variables are  $\xi^\circ$  and  $\eta^\circ$ . One must remember that  $\vec{\xi}^\circ$  occurs implicitly in  $\tau$ . The integration must be extended over the above mentioned strip. So far, the function  $h^\circ(\vec{\xi}^\circ, \tau)$  is unknown. One has, because  $z^\circ = 0$

$$\frac{\partial R_\circ}{\partial z^\circ} = \frac{z^\circ}{R_\circ}$$

therefore,

$$\phi^\circ(\vec{x}^\circ, t) = -\frac{z^\circ}{4\pi} \int \int \left( -\frac{h^\circ(\vec{\xi}^\circ, \tau)}{R_\circ^3} - \frac{h_r^\circ(\vec{\xi}^\circ, \tau)}{a_\circ R_\circ^2} \right) d\xi^\circ d\eta^\circ \quad (2.10)$$

where  $h_r^\circ$  denotes the derivative of  $h^\circ$  with respect to the second argument  $\tau$ . In potential theory one determines, for a sheet of sources, the normal component of the potential in terms of the local source intensity. The expression corresponding for the sources in the present case is given by Equation (2.7). Equation (2.9) is then the expression for the component of the potential gradient in the  $z$ -direction. We shall show by applying an analogous procedure that

$$\lim_{z^\circ \rightarrow +0} \phi^\circ(ix^\circ + jy^\circ + kz^\circ, t) = \lim_{z^\circ \rightarrow -0} \phi^\circ(ix^\circ + jy^\circ + kz^\circ, t) = h^\circ(ix^\circ + jy^\circ, t) \quad (2.11)$$

Here we repeat the main steps of the argument. Consider a point given by  $\vec{x}^\circ = ix^\circ + jy^\circ + kz^\circ$  and determine the limit of Eq.(2.10) as  $z^\circ \rightarrow +0$  while  $x^\circ$  and  $y^\circ$  are fixed. We divide the region of integration into an inner and outer region by a small circle with radius  $\rho$  around the point  $\xi^\circ = x^\circ, \eta^\circ = y^\circ$ . Outside the circle  $R_\circ \neq 0$ ; the integral (over the outside region) vanishes in the limit  $z^\circ \rightarrow 0$ . To carry out the integration within the small circle we introduce polar coordinates.

$$\xi^\circ - x^\circ = \rho \cos \theta$$

$$\eta^\circ - y^\circ = \rho \sin \theta$$

The dominant term of the integrand for small  $\rho_c$  is  $h^\circ(\vec{\xi}^\circ, t) \left( \frac{z^\circ}{R_c^3} \right)$  where  $\vec{\xi}^\circ = ix^\circ + jy^\circ$ . The contribution of this term is

$$\frac{1}{4\pi} h^\circ(ix^\circ + jy^\circ, t) \int_0^{\rho_c} \frac{2\pi z^\circ \rho}{(\rho^2 + z^{\circ 2})^{3/2}} d\rho = \frac{1}{2} h^\circ(ix^\circ + jy^\circ, t) \int_0^{\rho_c} \frac{z^\circ \rho}{(\rho^2 + z^{\circ 2})^{3/2}} d\rho$$

The integral is rewritten. Setting  $\rho/z^\circ = \tilde{\rho}$ , one obtains

$$\int_0^{\rho_c} \frac{z^\circ \rho}{(\rho^2 + z^{\circ 2})^{3/2}} d\rho = \int_0^{\tilde{\rho} = \rho_c/z^\circ} \frac{\tilde{\rho}}{(\tilde{\rho}^2 + 1)^{3/2}} d\tilde{\rho} = -(\tilde{\rho}^2 + 1)^{-1/2} \Big|_0^{\rho_c/z^\circ}$$

Hence

$$\lim_{z^\circ \rightarrow 0} \int_0^{\rho_c} \frac{z^\circ \rho}{(\rho^2 + z^{\circ 2})^{3/2}} d\rho = 1$$

With Equation (2.9), and the discussion following it, one now obtains

$$\lim_{z^\circ \rightarrow 0} \phi^\circ(ix^\circ + jy^\circ + kz^\circ, t) = \frac{1}{2} h^\circ(ix^\circ + jy^\circ, t)$$

Since the potential is antisymmetric one has

$$\lim_{z^\circ \rightarrow 0} \phi^\circ(ix^\circ + jy^\circ + kz^\circ, t) = -\frac{1}{2} h^\circ(ix^\circ + jy^\circ, t)$$

Accordingly  $h^\circ(ix^\circ + jy^\circ, t)$  represents the difference of the potential between the upper and the lower side of the doublet sheet at the point given by  $\vec{x}^\circ = (ix^\circ + jy^\circ)$ . The same result can be obtained by a simpler argument. According to the discussion leading to Eq.(2.9) the function  $\phi^\circ(\vec{x}^\circ, t)$  can be interpreted as the  $z^\circ$  component of the potential generated by a source sheet of strength  $h^\circ(ix^\circ + jy^\circ, t)$ . We showed, following Eq.(2.8), that the mass flow per unit time of a single source at a point  $\vec{\xi}^\circ$  is  $h^\circ(\vec{\xi}^\circ, t)$ . The mass flow per unit time emanating from an element  $dA$  surrounding the point  $\vec{\xi}^\circ = ix^\circ + jy^\circ$  is then  $h^\circ(\vec{\xi}^\circ, t)dA$ . The mass flow is also expressed by the difference of the  $z^\circ$  velocity at the upper and lower sides, multiplied by  $dA$ . But the velocity components are given by  $\lim_{z^\circ \rightarrow 0} \phi^\circ(ix^\circ + jy^\circ + kz^\circ, t)$  hence the result given in Eq.(2.11).

The motion of the wing has a  $z^\circ$  component of the velocity, which will be denoted  $w(x^\circ, y^\circ, t)$ . In the linearized approximation this is the velocity component normal to the wing and as such it is transmitted to the air. Therefore

$$w(x^\circ, y^\circ, t) = \lim_{z^\circ \rightarrow 0} \left( \frac{\partial \phi^\circ(\vec{x}^\circ, t)}{\partial z^\circ} \right) \quad (2.12)$$

where  $a$  is known on the wing surface

$$\frac{\partial \phi(\vec{r}, t)}{\partial z} = -\frac{1}{4\pi} \frac{\partial^2}{\partial z^2} \int \int_A \frac{1}{R} h^\circ(\vec{\xi}, \tau) d\xi^\circ d\eta^\circ \quad (2.13)$$

$$\tau = t - \frac{R}{a}$$

$$R = |\vec{r} - \vec{\xi}| = [(x - \xi)^2 + (y - \eta)^2 + z^2]^{\frac{1}{2}}$$

Furthermore

$$\frac{\partial R}{\partial z} = \frac{z}{R}$$

$$\frac{\partial \tau}{\partial z} = -\frac{z}{aR}$$

Carrying out the differentiation with respect to  $z$  we obtain

$$\frac{\partial \phi(\vec{r}, t)}{\partial z} = -\frac{1}{4\pi} \int \int \left[ \left( \frac{\partial^2}{\partial z^2} \frac{1}{R} \right) h^\circ(\vec{\xi}, \tau) + \frac{3z^2 - R^2}{aR^4} h_\tau^\circ(\vec{\xi}, \tau) + \frac{z^2}{a^2 R^3} h_{\tau\tau}^\circ(\vec{\xi}, \tau) \right] d\xi^\circ d\eta^\circ \quad (2.14)$$

where

$$\vec{\xi} = i\xi^\circ + j\eta^\circ$$

According to Equation (2.12) one must carry out the limiting process  $z^\circ \rightarrow 0$ . If the point  $(x, y)$  lies outside the region of integration, this can be done immediately and one obtains

$$\frac{\partial \phi(\vec{r}, t)}{\partial z} \Big|_{z=0} = \frac{1}{4\pi} \int \int \left[ \frac{1}{R^3} h^\circ(\vec{\xi}, t) + \frac{1}{aR^2} h_\tau^\circ(\vec{\xi}, \tau) \right] d\xi^\circ d\eta^\circ \quad (2.15)$$

This form of the equation will be useful in conjunction with the acceleration potential. There one encounters the task of evaluating this expression for points of the  $x, y$ -plane upstream of the wing.

For  $\vec{\xi} = ix^\circ + jy^\circ$ , one obtains  $R_z = 0$  and then it is not possible to carry out the limiting process  $z^\circ \rightarrow 0$  directly. Actually some contributions to the integral cancel each other. This is shown by means of certain transformations (in essence they are integrations by parts). We present here the approach of Professor Williams. He points out that  $\frac{1}{R}$  satisfies the Laplace equation in the  $x^\circ, y^\circ, z^\circ$  system, therefore;

$$\frac{\partial^2}{\partial z^2} \left( \frac{1}{R} \right) = - \left( \frac{\partial^2}{\partial x^2} \left( \frac{1}{R} \right) + \frac{\partial^2}{\partial y^2} \left( \frac{1}{R} \right) \right)$$

but as

$$R = [(x - \xi)^2 + (y - \eta)^2 + (z^\circ)^2]^{\frac{1}{2}}$$

One also has

$$\left( \frac{\partial^2}{\partial x^{c2}} + \frac{\partial^2}{\partial y^{c2}} \right) \left( \frac{1}{R_c} \right) = \left( \frac{\partial^2}{\partial \xi^{c2}} + \frac{\partial^2}{\partial \eta^{c2}} \right) \left( \frac{1}{R_c} \right) = \nabla_2^{c2} \left( \frac{1}{R_c} \right)$$

with

$$\nabla_2^c \hat{=} i \frac{\partial}{\partial \xi^c} + j \frac{\partial}{\partial \eta^c}$$

The superscript  $c$  indicates that the del operator is defined in the  $\xi^c, \eta^c$ -system (as opposed to  $\xi, \eta$ -system); the subscript 2 emphasizes that it is defined in the two-dimensional  $\xi^c, \eta^c$ -space (rather than in the  $\xi^c, \eta^c, \zeta^c$ -space). The first term in the integrand of Equation (2.14) therefore assumes the form

$$h^c(\vec{\xi}^c, \tau) \frac{\partial^2}{\partial z^{c2}} \left( \frac{1}{R_c} \right) = -h^c(\vec{\xi}^c, \tau) \left[ \nabla_2^{c2} \left( \frac{1}{R_c} \right) \right]$$

This is rewritten by means of the following relation, easily verified if one writes it out in components. Let  $a(\xi^c, \eta^c)$  be a scalar field and  $\vec{v}(\xi^c, \eta^c)$  a vector field. Then one has

$$a \nabla_2^c \cdot \vec{v} = \nabla_2^c \cdot (a \vec{v}) - \nabla_2^c a \cdot \vec{v}$$

Accordingly

$$h^c(\vec{\xi}^c, \tau) \frac{\partial^2}{\partial z^{c2}} \left( \frac{1}{R_c} \right) = -\nabla_2^c \cdot \left( h^c(\vec{\xi}^c, \tau) \nabla_2^c \left( \frac{1}{R_c} \right) \right) + \nabla_2^c \left( \frac{1}{R_c} \right) \cdot \nabla_2^c h^c(\vec{\xi}^c, \tau)$$

In forming  $\nabla_2^c h^c(\vec{\xi}^c, \tau)$  one will remember that the vector  $\vec{\xi}^c$  occurs in  $h^c$  as the first argument and also implicitly in  $\tau = t - \frac{R_c}{a_0}$ . To make clear which partial derivatives are meant, one best writes  $h^c(\vec{\xi}^c, \tau) = h^c(\xi^c, \eta^c, \tau)$  and applies the usual notation for partial derivatives. Then

$$\nabla_2^c(h(\xi^c, \eta^c, \tau)) = i h_{\xi^c}^c + j h_{\eta^c}^c + h_{\tau} \nabla_2^c \tau$$

(even for  $z^c \neq 0$ ). Let

$$\vec{\gamma}(\vec{\xi}^c, \tau) \hat{=} i h_{\xi^c}^c + j h_{\eta^c}^c \quad (2.16)$$

and

$$\vec{\rho} = i(\xi^c - x^c) + j(\eta^c - y^c) \quad (2.17)$$

then even for  $z^c \neq 0$

$$\nabla_2^c R_c = \frac{\vec{\rho}}{R_c}$$

and

$$\nabla_2 \left( \frac{1}{R_c} \right) = -\frac{\vec{\rho}}{R_c^3} \quad (2.18)$$

and by Equation (2.8)

$$\nabla_2 \vec{\tau} = -\frac{\vec{\rho}}{R_c a_c}$$

Therefore, with  $\vec{\rho} \cdot \vec{\rho} = |\vec{\rho}|^2 = (R^2 - z^2)$

$$h(\vec{\xi}, \tau) \frac{\partial^2}{\partial z^2} \left( \frac{1}{R} \right) = -\nabla_2 \cdot \left( h(\vec{\xi}, \tau) \nabla_2 \left( \frac{1}{R_c} \right) \right) - \frac{\vec{\rho}}{R^3} \cdot \vec{\gamma}(\vec{\xi}, \tau) + h_{\tau}(\vec{\xi}, \tau) \frac{R_c^2 - z^2}{a_c R_c^4}$$

Substituting this expression into Equation (2.14) one obtains:

$$w(\vec{\xi}, t) = \frac{-1}{4\pi} \int \int_{A_c} \left[ -\nabla_2 \cdot \left( h(\vec{\xi}, \tau) \nabla_2 \left( \frac{1}{R_c} \right) \right) - \left( \frac{\vec{\rho}_c}{R_c^3} \right) \cdot \vec{\gamma}(\vec{\xi}, \tau) + \left( \frac{2z^2}{a_c R_c^4} \right) h_{\tau}(\vec{\xi}, \tau) + \frac{z^2}{a_c^2 R_c^3} h_{\tau\tau}(\vec{\xi}, \tau) \right] d\xi^\circ d\eta^\circ$$

Notice that these transformations hold for  $z^\circ \neq 0$ . Thus  $R_c \neq 0$  and the expression on the left in the following equation can be transformed by the Gaussian integral theorem.

$$\int \int_A -\nabla_2 \cdot \left[ h(\vec{\xi}, \tau) \nabla_2 \left( \frac{1}{R_c} \right) \right] d\xi^\circ d\eta^\circ = - \int_S h(\vec{\xi}, \tau) \nabla_2 \left( \frac{1}{R_c} \right) \cdot \vec{n} ds \quad (2.19)$$

where  $\vec{n}$  is the unit vector in the direction of the outer normal to the boundary of the area of integration,  $ds$  is the line element and  $S$  is the contour of the region  $A_c$ . But we found that outside of the area of integration  $\phi(\vec{\xi}^\circ, t) = 0$  if  $\vec{\xi}^\circ$  lies in the  $x^\circ, y^\circ$  plane and that  $\vec{h}$  is the jump of  $\phi$  at the plane  $z^\circ = 0$ ; there is no jump of  $\phi^\circ$  outside the strip of the  $x^\circ, y^\circ$  plane swept out by the wing. Moreover,  $\phi^\circ$  is continuous, as one moves across the boundary of the region  $A$  (otherwise there would arise a delta function in the velocity). Therefore  $h(\vec{\xi}, t) = 0$  at the boundary of  $A_c$  and the expression shown in Equation (2.19) vanishes. This procedure eliminates the worst of the singularities. Equation (2.12) is now written in the form

$$w(\vec{x}, t) = w_v(\vec{x}, t) + w_p(\vec{x}, t) \quad (2.20)$$

where

$$w_v(\vec{x}, t) = \frac{1}{4\pi} \int \int_{A_c} \frac{\vec{\rho}}{R^3} \cdot \vec{\gamma}(\vec{\xi}, \tau) d\xi^\circ d\eta^\circ \quad (2.21)$$

and

$$w_p(\vec{x}, t) = \lim_{z \rightarrow 0} \frac{1}{4\pi} \int \int_A \frac{z^2}{a_c R^3} \left[ \frac{2}{R_c} h_{\tau}(\vec{\xi}, \tau) + \frac{1}{a_c} h_{\tau\tau}(\vec{\xi}, \tau) \right] d\xi^\circ d\eta^\circ \quad (2.22)$$

provided that  $h(\vec{\xi}, \tau) \equiv 0$  at the boundary of the region  $A_c$ . The expression  $\vec{\gamma} = ih_{\xi^c}^c + jh_{\eta^c}^c$  is defined in Equation (2.16) and can be interpreted as follows. We found that  $\omega(\xi, \eta, \tau) = \frac{1}{2}h(\xi, \eta, \tau)$ , therefore  $\frac{1}{2}(ih_{\xi^c}^c + jh_{\eta^c}^c)$  gives the velocity component on the surface at the upper side of the sheet and  $-\frac{1}{2}(ih_{\xi^c}^c + jh_{\eta^c}^c)$  is the velocity at the lower side. The strip on which  $h \neq 0$  is therefore a shear layer, which can be interpreted as a vortex sheet (Hence the subscript v). The two velocity components have opposite sign. The vorticity vector lies in the  $\xi, \eta$  plane and is perpendicular to the vector  $(ih_{\xi^c}^c + jh_{\eta^c}^c)$ . The expression  $\frac{\vec{\rho} \cdot \vec{\gamma}(\vec{\xi}, \tau)}{R_c^3}$  can be obtained by the Biot-Savart law. Not much is gained by this interpretation. An isolated vortex element does not exist, because it violates  $\text{div curl } \vec{v} = 0$ . An infinitesimal vortex ring, according to Stokes' vorticity law, is equivalent to a doublet distribution over a surface which has the vortex ring as boundary, and this brings us back to the starting point of these investigations.

For  $z^c \rightarrow 0$ , and  $\vec{\xi}^c \rightarrow \vec{x}^c$ , the distance  $R_c \rightarrow 0$  and one still encounters a singularity because of the denominator  $R_c^3$  in Equation (2.21). We rewrite

$$u_v(\vec{x}, t) = \frac{1}{4\pi} \int \int \vec{\gamma}(\vec{\xi}, \tau) \cdot \frac{\vec{\rho}}{R_c^3} d\xi^c d\eta^c =$$

$$\frac{1}{4\pi} \vec{\gamma}(ix^c + jy^c, t) \cdot \int \int \frac{\vec{\rho}}{R_c^3} d\xi^c d\eta^c + \frac{1}{4\pi} \int \int (\vec{\gamma}(\vec{\xi}, \tau) - \vec{\gamma}(ix^c + jy^c, \tau)) \cdot \frac{\vec{\rho}}{R_c^3} d\xi^c d\eta^c \quad (2.23)$$

In the second term the singularity vanishes. This becomes evident if the integration is carried out in polar coordinates:

$$\xi^c - x^c = \rho \cos \theta$$

$$\eta^c - y^c = \rho \sin \theta$$

$$d\xi^c d\eta^c \rightarrow \rho d\rho d\theta$$

where

$$\rho = |\vec{\rho}|$$

Now  $(\vec{\gamma}(\vec{\xi}, \tau) - \vec{\gamma}(ix^c + jy^c, \tau))$  is  $O(\rho)$ . We therefore obtain an integrand  $\frac{O(\rho^3)}{(\rho^2 + z^2)^{3/2}} d\theta$ , which, in the limit  $z \rightarrow 0$ , becomes  $O(1)d\theta$ . In the first integral in Equation (2.23), the introduction of polar coordinates would give  $\frac{\rho \vec{\rho}}{(\rho^2 + z^2)^{3/2}} d\rho d\theta$ . Carrying out the limiting process  $z \rightarrow 0$  at this stage, one obtains  $O\left(\frac{1}{\rho}\right) d\rho$ , which does not converge for  $\rho = 0$ .

For  $z \neq 0$  this integral is rewritten using Equation (2.18) and subsequently the gradient theorem.

$$\int \int \frac{\bar{\rho}}{R_c^3} d\xi^\circ d\eta^\circ = - \int \int \nabla_\perp \left( \frac{1}{R_c} \right) d\xi^\circ d\eta^\circ = \oint \frac{1}{R_c} \bar{n} ds$$

where  $\bar{n}$  is the unit vector in the direction of the outer normal and  $ds$  is the arc length of the perimeter of the region under consideration. At the boundary curve, along which the integration on the right is carried out,  $R_c \neq 0$ , so that one can perform the limiting process  $z \rightarrow 0$ , then  $R \rightarrow |\bar{\rho}|$ . The gradient theorem is readily derived by carrying out the integrations in the following expression

$$\int \int \left( i \frac{\partial u}{\partial x} + j \frac{\partial u}{\partial y} \right) dx dy = \oint u (i dy - j dx) = \oint u \bar{n} ds$$

In evaluating  $w_p$ , Equation (2.22), in the limit  $z^\circ \rightarrow 0$ , a technique familiar from potential theory is applied again. Consider a point given by

$$\bar{x}^\circ = x^\circ i + y^\circ j + z^\circ k$$

By drawing a small circle with radius  $\rho_c$  around the point  $\xi^\circ = x^\circ, \eta^\circ = y^\circ$  of the  $\xi^\circ, \eta^\circ$ -plane, we divide the region of integration into an inner and an outer region. In the outer region  $R_c \neq 0$ , and the limiting process  $z^\circ \rightarrow 0$  gives zero immediately. If  $\rho_c$  is sufficiently small and for  $z^\circ \rightarrow 0$ ,  $h_\tau^\circ(\xi^\circ, \tau)$  can be replaced by  $h_t^\circ(ix^\circ + jy^\circ, t) = h_t^\circ(ix^\circ + jy^\circ, t)$ . Thus one obtains as contribution of the first term in Equation (2.22) for  $w_p$

$$-\frac{1}{2\pi} h_t^\circ(ix^\circ + jy^\circ, t) \int \int \frac{z^{\circ 2}}{a_c R_c^4} d\xi d\eta$$

In polar coordinates with polar radius  $\rho$ , one obtains

$$-\frac{h_t(x, y, t)}{a_c} \int_0^{\rho_c} \frac{z^{\circ 2} \rho}{(\rho^2 + z^{\circ 2})^2} d\rho$$

Hence with  $(\rho/z^\circ)^2 = u$

$$\begin{aligned} -\frac{h_t(x, y, t)}{a} \int_0^{\rho_c} \frac{z^{\circ 2} \rho}{(\rho^2 + z^{\circ 2})^2} d\rho &= -\frac{h_t(x, y, t)}{2a_c} \int_{u=0}^{u=(\frac{\rho_c}{z^\circ})^2} \frac{du}{(u+1)^2} \\ &= \frac{h_t(x, y, t)}{2a_c} \left( \frac{1}{(u+1)} \right)_{u=0}^{u=(\frac{\rho_c}{z^\circ})^2} \end{aligned}$$

Then

$$\lim_{z^\circ \rightarrow 0} -\frac{h_t(x, y, t)}{2\pi a_c} \int \int_0^{\rho_c} \frac{z^{\circ 2} \rho d\theta d\rho}{(\rho^2 + z^{\circ 2})^2} = 0 - \frac{h_t(ix^\circ + jy^\circ, t)}{2a}$$

Treating the second term in Equation (2.22) in a similar manner: one obtains zero. Thus

$$w_p(\tilde{x}^c, t) = -\frac{h_t(\tilde{x}^c, t)}{2a_+}$$

For an interpretation of this term we observe the following. If  $w(x^c, y^c, +0, t)$  does not depend upon  $x^c$  and  $y^c$  then the same holds for  $h(x^c, y^c, t)$ . The integral equation then reduces to  $w(t) = \frac{1}{2a_+} \frac{dh}{dt}$ . As  $w(t) = \phi_z^c(t)$ , and  $\frac{h}{2} = \phi(x^c, y^c, +0, t)$ , one then obtains  $\phi_z^c = -\frac{1}{a_+} \phi_t^c$ . This relation holds for a plane wave propagating in the  $z^c$  direction. This is the approximation of piston theory (therefore the subscript p).

In the acoustical approximation, the perturbation pressures are expressed (from the Bernoulli equation for unsteady flow) by

$$\Delta p = -\rho_\infty \phi_t = -\frac{\rho_\infty}{2} h_t^c(\xi^c, \eta^c, +0, t)$$

where  $\rho_\infty$  denotes the density of the undisturbed air. The pressure difference between the upper and the lower side of the wake is 0. Therefore  $h_t = 0$  at points of the wake. In the  $x^c, y^c, z^c$  system the value of  $h^c$  remains constant for points of the wake. It is determined by the value it assumed at the moment when the trailing edge swept over this point.

In the present formulation it has been assumed that the wing moves within the  $x^c, y^c$ -plane. It need not move in a straight line or with constant velocity. For such a situation, or even if the wing motion is not restricted to this plane, the  $x^c, y^c, z^c$ -system is the appropriate system of coordinates. (It is, of course assumed that the motions are compatible with linearized theory.) In the more general cases one will have a wake, which does not lie in the  $x^c, y^c$  plane but the condition  $\frac{\partial h^c}{\partial t} = 0$  for points of the wake still applies. If the wing does not lie within the  $x^c, y^c$  plane, then one must introduce doublets oriented in the direction normal to the wing (rather than in the  $z^c$  direction). The wing motion and deformation always determines the velocity component in the direction normal to the wing.



### Section III

#### Wing Coordinates

For a wing moving with constant speed  $U$  in the  $x$ -direction, it is convenient to transform the equations to a Cartesian system fixed with respect to the wing. Accordingly we set

$$x = x' + U't$$

$$y = y'$$

$$z = z'$$

$$t = t'$$

The manner in which  $\xi$  (and  $\eta$ ) is transformed into wing coordinates is based on the following reflection. The function  $h$  appears in the integral equation with a third argument  $\tau$ . We want a newly defined function  $h(\xi, \eta, \tau)$  to refer to a fixed location on the wing. Then the relation between  $\xi$  and  $\xi'$  is determined by the location of the wing with respect to the  $\xi', \eta'$  system at the time given by  $\tau$ . Therefore we have

$$\xi = \xi' + U\tau$$

$$\eta = \eta'$$

$$h(\xi, \eta, \tau) = h'(\xi - U\tau, \eta, \tau)$$

As far as the integral is concerned this is a transformation from the variables  $\xi'$  and  $\eta'$  to  $\xi$  and  $\eta$ . In this transformation  $x, y, z$  and  $t$  play the role of parameters, they enter through the definition of the auxiliary function  $\tau$ .

$$\tau = t - R/a \tag{3.1}$$

with

$$R = \left[ (\xi' - x)^2 + (\eta' - y)^2 + z^2 \right]^{1/2} \tag{3.2}$$

In at least one expression it is simpler to form the limit  $z \rightarrow 0$  after the transformation has been carried out. The transformation from  $\xi', \eta'$  to  $\xi, \eta$  is straightforward. The inverse transformation is less so. We introduce

$$M = U/a$$

then

$$(\xi^\circ - x^\circ) = (\xi - x) + U(t - \tau)$$

Hence with Equation (3.1)

$$(\xi^\circ - x^\circ) = (\xi - x) + MR_c \quad (3.3)$$

substituting Equation (3.3) into Equation (3.2) one obtains

$$R_c^2 = ((\xi - x) + MR_c)^2 + (\eta - y)^2 + z^2$$

This is a quadratic equation for  $R_c$ . Writing it in detail one has

$$R_c^2(1 - M^2) - 2M(\xi - x)R_c = (\xi - x)^2 + (\eta - y)^2 + z^2$$

The result is written in the following form. Let

$$\beta^2 = 1 - M^2 \quad (3.4)$$

and

$$R_c = [(\xi - x)^2 + \beta^2(\eta - y)^2 + \beta^2 z^2]^{\frac{1}{2}} \quad (3.5)$$

then

$$R_c = \frac{1}{\beta^2} [M(\xi - x) + R_c] \quad (3.6)$$

From now on  $R_c$  and also  $\tau$  are considered as functions of  $x, y, \xi, \eta$ , and  $t$ . Substituting Equation (3.6) into equation (3.1), one obtains  $\tau$ ; subsequently one finds  $\xi^\circ$ . The transformation of  $\eta^\circ$  and  $z^\circ$  are trivial. According to Equation (2.17)

$$\vec{\rho} = i(\xi^\circ - x^\circ) + j(\eta^\circ - y^\circ)$$

Then with Equation (3.3) in terms of the new coordinates

$$\vec{\rho} = i(\xi - x + MR_c) + j(\eta - y) \quad (3.7)$$

Substituting  $R_c$  one obtains

$$\vec{\rho} = \frac{i}{\beta^2}(\xi - x + MR_c) + j(\eta - y) \quad (3.8)$$

One obtains from Equations (3.5) and (3.6)

$$\frac{\partial R_c}{\partial \xi} = \frac{1}{\beta^2} \frac{1}{R_c} [MR_c + \xi - x] \quad (3.9)$$

Furthermore

$$\frac{\partial R_c}{\partial \eta} = \frac{\eta - y}{R_c} \quad (3.10)$$

Let

$$\nabla_2 \triangleq i \frac{\partial}{\partial \xi} + j \frac{\partial}{\partial \eta} \quad (3.11)$$

Then because of Equation (3.8)

$$\nabla_2 R_c = \frac{\bar{\rho}}{R_c} \quad (3.12)$$

$$\nabla_2 \frac{1}{R_c} = -\frac{\bar{\rho}}{R_c^2 R_c} \quad (3.13)$$

The surface element  $dA = d\xi^\circ d\eta^\circ$  transforms into  $\left(\frac{\partial \xi^\circ}{\partial \xi}\right) d\xi d\eta$ . One obtains from Equation (3.3)

$$\frac{\partial \xi^\circ}{\partial \xi} = 1 + M \frac{\partial R_c}{\partial \xi}$$

Hence with Equation (3.9)

$$\frac{\partial \xi^\circ}{\partial \xi} = \frac{1}{R_c} (R_c + M^2 R_c + M(\xi - x))$$

and with Equation (3.6)

$$\frac{\partial \xi^\circ}{\partial \xi} = \frac{R_c}{R_c}$$

Thus  $dA = (R_o/R_c) d\xi d\eta$  Let

$$h(\xi, \eta, \tau) = h^\circ(\xi - U\tau, \eta, \tau) \quad (3.14)$$

Using Equation (2.16)

$$\vec{\gamma}(\xi, \eta, \tau) = ih_\xi + jh_\eta = ih_\xi^\circ + jh_\eta^\circ = \vec{\gamma}(\xi^\circ, \eta^\circ, \tau) \quad (3.15)$$

where  $\tau$  is obtained from Equation (3.1)

$$\tau = t - \frac{R_o}{a_o} = t - \frac{1}{a_o \beta^2} [M(\xi - x) + R_c] \quad (3.16)$$

and  $R_c$  is given in Equation (3.5). According to the definition of  $h$ , Equation (3.14),

$$h(x, y, t) = h^\circ(x - Ut, y, t)$$

then

$$h_r(x, y, t) = h_{x^\circ}^\circ(x - Ut, y, t) = h_{x^\circ}^\circ(x^\circ, y^\circ, t)$$

$$h_t(x, y, t) = -U h_{x^\circ}^\circ(x - Ut, y, t) + h_t^\circ(x - Ut, y, t)$$

$$h_t(x, y, t) = -U h_{x^\circ}^\circ(x^\circ, y^\circ, t) + h_t^\circ(x^\circ, y^\circ, t)$$

Therefore

$$h_t^\circ(x^\circ, y^\circ, t) = h_t(x, y, t) + U h_x(x, y, t)$$

One thus obtains the integral equation

$$w(x, y, 0, t) = w_v(x, y, 0, t) + w_p(x, y, 0, t) \quad (3.17)$$

where

$$w_p(x, y, 0, t) = -\frac{1}{2a_c} (h_t(x, y, t) + U h_x(x, y, t)) \quad (3.18)$$

$$w_v(x, y, 0, t) = \frac{1}{4\pi} \iint \frac{\vec{\rho} \cdot \vec{\gamma}(\xi, \eta, \tau)}{R_o^2 R_c} d\xi d\eta \quad (3.19)$$

$$\vec{\gamma} \triangleq i h_\xi + j h_\eta$$

The vector  $\vec{\rho}$  is found in Equation (3.7). When we treated the problem in the  $x^\circ, y^\circ, z^\circ$  system, the expression  $w_v$  had been decomposed in order to deal with the dominant singularity of the integrand by analytical means. The same decomposition is carried out here.

$$\begin{aligned} w_v = \frac{1}{4\pi} \lim_{z \rightarrow 0} \iint \frac{\vec{\gamma}(\xi, \eta, \tau) \cdot \vec{\rho}}{R_o^2 R_c} d\xi d\eta &= \frac{1}{4\pi} \vec{\gamma}(x, y, t) \cdot \iint \frac{\vec{\rho}}{R_o^2 R_c} d\xi d\eta \\ &+ \frac{1}{4\pi} \iint \frac{[\vec{\gamma}(\xi, \eta, \tau) - \vec{\gamma}(x, y, t)] \cdot \vec{\rho}}{R_o^2 R_c} d\xi d\eta \end{aligned} \quad (3.20)$$

In the first term a limiting process  $z \rightarrow 0$  is needed. According to Equation (3.13)

$$\frac{\vec{\rho}}{R_o^2 R_c} = -\nabla_2 \left( \frac{1}{R_o} \right)$$

then

$$\iint \frac{\vec{\rho} d\xi d\eta}{R_o^2 R_c} = - \iint \nabla_2 \left( \frac{1}{R_o} \right) d\xi d\eta = - \oint \frac{1}{R_o} \vec{n} ds \quad (3.21)$$

according to the gradient theorem. In the right hand side of Equation (3.21) one is permitted to make the limiting process  $z \rightarrow 0$ , i.e., one replaces  $R_o(\xi, \eta, z)$  by  $R_o(\xi, \eta, 0)$ . The postponement of the limiting process  $z \rightarrow 0$  has been necessary, because, in transforming from  $\xi^\circ, \eta^\circ$  to  $\xi, \eta$ , the contour of the region will in general not retain its shape; the normal in the  $\xi^\circ, \eta^\circ$  system is different from the normal in the  $\xi, \eta$  system and so is the length of the line element. The result, Equation (3.21), is now substituted into Equation (3.20).

$$w_v(x, y, 0, t) = \frac{-1}{4\pi} \vec{\gamma}(x, y, t) \cdot \oint \frac{1}{R_c} \vec{n} ds + \frac{1}{4\pi} \iint \frac{[\vec{\gamma}(\xi, \eta, \tau) - \vec{\gamma}(x, y, t)] \cdot \vec{\rho}}{R_o^2 R_c} d\xi d\eta \quad (3.22)$$

Note that Eq.(3.22) is identical to equations (VI.18) and (VI.19) of Reference 1. The transition from one formulation to the other is shown in Appendix.

The effect of the change of the shape of an area of integration connected with this transformation is also observed if one considers the expression for the upwash in its original form, i.e., before the transformations to remove the singularities are carried out. This expression has meaning if  $x$  and  $y$  do not lie within the region of integration. One has for the potential of a source sheet in the  $x, y, \xi, \eta$ -system

$$\phi^s(x, y, t) = \frac{-1}{4\pi} \iint \frac{h(\xi, \eta, \tau) d\xi d\eta}{R_c}$$

with

$$\tau = t - \frac{R_c}{a}$$

Hence in the  $x, y, \xi, \eta$  systems

$$\phi^s(x, y, t) = \frac{-1}{4\pi} \iint \frac{h(\xi, \eta, \tau) d\xi d\eta}{R_c} \quad (3.23)$$

As before

$$h(\xi, \eta, \tau) = h(\xi - U\tau, \eta, \tau) \quad (3.24)$$

$$\tau = t - \frac{1}{a\beta^2} [M(\xi - x) + R_c] \quad (3.25)$$

$$R_c = [(\xi - x)^2 + \beta^2(\eta - y)^2 + \beta^2 z^2]^{\frac{1}{2}} \quad (3.26)$$

Since

$$\frac{dR_c}{dz} = \frac{\beta^2 z}{R_c}$$

one obtains in the limit  $z \rightarrow 0$

$$\phi_z^d(x, y, 0, t) = \phi_{zz}^s(x, y, 0, t) = \frac{1}{4\pi} \iint \left[ \beta^2 \frac{h(\xi, \eta, \tau)}{R_c^3} + \frac{h_\tau(\xi, \eta, \tau)}{a R_c^2} \right] d\xi d\eta \quad (3.27)$$

Alternatively one can begin by forming the derivatives with respect to  $z$  in the  $x, y, \xi, \eta$  systems,

$$\phi_{zz}^d(x, y, t) = \frac{1}{4\pi} \iint \left[ \frac{h(\xi, \eta, \tau)}{R^3} + \frac{h_\tau(\xi, \eta, \tau)}{a R^2} \right] d\xi d\eta$$

In the subsequent transformation to the  $\xi, \eta$  system one has

$$h(\xi, \eta, \tau) = h(\xi + U\tau, \eta, \tau)$$

For the sake of clarity we denote the derivatives with respect to the first and third argument not by  $h_\xi$  and  $h_\tau$  but by  $h^{(1)}$  and  $h^{(3)}$ . The transformation then gives the alternative expression

$$h_\tau^\tau = U h^{(1)} + h^{(3)}$$

therefore

$$\phi_z^d(x, y, 0, t) = \frac{1}{4\pi} \int \int \left[ \frac{h(\xi, \eta, \tau)}{R_o^2 R_c} + \frac{U h^{(1)}(\xi, \eta, \tau) + h^{(3)}(\xi, \eta, \tau)}{a_o R_c R_c} \right] d\xi d\eta \quad (3.28)$$

To bring Equations (3.27) and (3.28) into agreement an integration by parts must be carried out. We are solely concerned with the integration with respect to  $\xi$ , while  $\eta$  is kept constant. One notices that  $\frac{dh}{d\xi}$  for  $\eta = \text{constant}$  is not identical with  $h^{(1)}$  because  $\xi$  occurs also in  $\tau$ . We express  $\frac{\partial \tau}{\partial \xi}$  in the following equation by means of Equations (3.25) and (3.26).

$$\frac{dh}{d\xi} = h^{(1)} + h^{(3)} \frac{\partial \tau}{\partial \xi} = h^{(1)} - h^{(3)} \frac{1}{a_o \beta^2} \frac{M R_c + \xi - x}{R_c}$$

With this equation we rewrite the numerator of the second term in Equation (3.28)

$$\begin{aligned} U h^{(1)} + h^{(3)} &= U \frac{dh}{d\xi} + h^{(3)} \left( 1 + \frac{M}{1 - M^2} \frac{M R_c + \xi - x}{R_c} \right) \\ U h^{(1)} + h^{(3)} &= U \frac{dh}{d\xi} + \frac{h^{(3)}}{(1 - M^2) R_c} (R_c + M(\xi - x)) \end{aligned}$$

and with Equation (3.6)

$$U h^{(1)} + h^{(3)} = U \frac{dh}{d\xi} + \frac{h^{(3)} R_o}{R_c}$$

Now Equation (3.28) appears in the form

$$\phi_z^d(x, y, 0, t) = \frac{1}{4\pi} \int \int \left[ \frac{h(\xi, \eta, \tau)}{R_o^2 R_c} + \frac{M \frac{dh}{d\xi}}{R_o R_c} + \frac{h^{(3)}(\xi, \eta, \tau)}{a_o R_c^2} \right] d\xi d\eta$$

An integration by parts is applied to the second term.

$$\phi_z^d(x, y, 0, t) = \frac{1}{4\pi} \left[ \oint \frac{M h}{R_c R_c} d\eta + \int \int h(\xi, \eta, \tau) \left( \frac{1}{R_o^2 R_c} - M \frac{\partial}{\partial \xi} \frac{1}{(R_o R_c)} + \frac{h^{(3)}}{a_o R_c^2} \right) d\xi d\eta \right] \quad (3.29)$$

Now

$$\begin{aligned} R_o R_c &= \frac{1}{1 - M^2} [M(\xi - x) R_c + R_c^2] \\ - \frac{\partial}{\partial \xi} \left( \frac{1}{R_o R_c} \right) &= \frac{1}{(1 - M^2) R_o^2 R_c^2} \left[ M R_c + \frac{M(\xi - x)^2}{R_c} + 2(\xi - x) \right] \end{aligned}$$

Then one obtains for the factor of  $h$  in Equation (3.29)

$$\frac{1}{R_c^2 R_c^2} \left[ R_c + \frac{M}{1-M^2} (M R_c + \frac{M(\xi-x)^2}{R_c} + 2(\xi-x)) \right] =$$

$$\frac{1}{(1-M^2)} \frac{1}{R_c^2 R_c^3} [R_c^2 + 2M(\xi-x)R_c + M^2(\xi-x)^2] = (1-M^2) \frac{1}{R_c^3}$$

Thus

$$\phi_z^d(x, y, 0, t) = \frac{1}{4\pi} \left[ \oint \frac{Mh}{R_c R_c} d\eta + \int \int \left[ \frac{\beta^2 h}{R_c^3} + \frac{h_\tau}{a_c R_c^2} \right] d\xi d\eta \right]$$

This agrees with Equation (3.27) except for the contour integral. If  $h = 0$  at the contour, this integral vanishes. The formula is applicable even if  $h \neq 0$  on the boundary, in which case the term accounts for the deformation of the region which occurs as one moves from the  $\xi, \eta$  system to the  $\xi, \eta$  system.

## Section IV

### Lorentz Coordinates

As before we denote  $x, y, z$  as a Cartesian system at rest with respect to the undisturbed air, and with  $t$  (formerly  $t$ ) the time. The transition to wing coordinates has been made by setting

$$x = x' + Ma' t' \quad (4.1)$$

$$y = y'$$

$$z = z'$$

$$t = t'$$

The Lorentz transformation<sup>2</sup> is given by

$$\hat{x} = (x' + Ma' t')/\beta \quad (4.2)$$

$$\hat{y} = y'$$

$$\hat{z} = z'$$

$$a' \hat{t} = (a' t' + Mx')/\beta$$

with the inverse

$$x' = (\hat{x} - Ma' \hat{t})/\beta \quad (4.3)$$

$$y' = \hat{y}$$

$$z' = \hat{z}$$

$$a' t' = (a' \hat{t} - M\hat{x})/\beta$$

It follows from the first of Equations (4.1) and (4.2) that

$$\hat{x} = x/\beta \quad (4.4)$$

$$\hat{y} = y$$

$$\hat{z} = z$$

Since Equations (4.4) do not contain  $t$ , the  $\hat{x}, \hat{y}, \hat{z}$ -system is fixed with respect to the  $x, y, z$ -system. The time  $t' = t$  is expressed in terms of  $\hat{t}$  and  $\hat{x}$  by the last of Equations (4.3). Substituting here  $\hat{x} = x/\beta$  one obtains

$$a' \hat{t} = a' t/\beta + M \frac{x}{\beta} \quad (4.5)$$



Equation (4.4) and (4.5) give the relation between the wing system  $x, y, z, t$  and the Lorentz system  $\hat{x}, \hat{y}, \hat{z}, \hat{t}$ . One has as inverse transformation

$$\begin{aligned}x &= \beta \hat{x} \\y &= \hat{y} \\z &= \hat{z} \\a \cdot t &= \frac{(a \cdot \hat{t} - M \hat{x})}{\beta}\end{aligned}$$

As mentioned above, the Lorentz transformation leaves the wave equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} - \frac{1}{a^2} \frac{\partial^2 \psi}{\partial t^2} = 0$$

unchanged, i.e. one obtains in the  $\hat{x}, \hat{y}, \hat{z}, \hat{t}$  system

$$\frac{\partial^2 \psi}{\partial \hat{x}^2} + \frac{\partial^2 \psi}{\partial \hat{y}^2} + \frac{\partial^2 \psi}{\partial \hat{z}^2} - \frac{1}{a_c^2} \frac{\partial^2 \psi}{\partial \hat{t}^2} = 0$$

The integral equation in the  $\hat{x}, \hat{y}, \hat{z}, \hat{t}$  system therefore has the same form as in the  $x, y, z, t$ -system, namely

$$\begin{aligned}\hat{u}(\hat{x}, \hat{y}, \hat{t}) &= \frac{\partial \hat{\phi}}{\partial \hat{z}}(\hat{x}, \hat{y}, \hat{z} = 0, \hat{t}) \\&= \frac{1}{4\pi} \int \int \frac{\vec{\rho} \cdot \vec{\gamma}}{\hat{R}^3} d\hat{\xi} d\hat{\eta} - \frac{1}{2a} \frac{\partial \hat{h}(\hat{x}, \hat{y}, \hat{t})}{\partial \hat{t}}\end{aligned}$$

where

$$\begin{aligned}\vec{R} &= (\hat{\xi} - \hat{x})\hat{i} + (\hat{\eta} - \hat{y})\hat{j} \\ \hat{R} &= |\vec{R}| \\ \vec{\gamma} &\triangleq \hat{i}\hat{h}_{\hat{\xi}}(\hat{\xi}, \hat{\eta}, \tau) + \hat{j}\hat{h}_{\hat{\eta}}(\hat{\xi}, \hat{\eta}, \tau)\end{aligned}$$

In this equation  $\hat{h}(\hat{x}, \hat{y}, \hat{t})$  is the unknown function. The transformation gives

$$h(x, y, t) = \hat{h}(\hat{x}, \hat{y}, \hat{t}) = \hat{h}\left(\frac{x}{\beta}, y, \frac{1}{a} \left(\beta a \cdot t + \frac{Mx}{\beta}\right)\right)$$

and

$$\phi(x, y, z, t) = \hat{\phi}(\hat{x}, \hat{y}, \hat{z}, \hat{t}) = \hat{\phi}\left(\frac{x}{\beta}, y, z, \frac{1}{a} \left(\beta a \cdot t + \frac{Mx}{\beta}\right)\right)$$

Hence

$$\phi_z(x, y, z, t) = \hat{\phi}_z(\hat{x}, \hat{y}, \hat{z}, \hat{t})$$

If  $\phi$  is the velocity potential, then in most practical applications

$$\phi_z(x, y, 0, t) \stackrel{\Delta}{=} w(x, y, t)$$

is a given function, on the wing. Then

$$\hat{w}(\hat{x}, \hat{y}, \hat{t}) = \hat{\phi}_z(\hat{x}, \hat{y}, 0, \hat{t}) = w(\beta \hat{x}, \hat{y}, (a_c \hat{t} - M \hat{x}) / (a_c \beta))$$

In the indicial problem one has for instance

$$w(x, y, t) = g(x, y)H(t)$$

where the step function  $H(t)$  is defined by

$$H(t) = 1, \quad t > 0$$

$$H(t) = 0, \quad t < 0$$

Then

$$\hat{w}(\hat{x}, \hat{y}, t) = g(\beta \hat{x}, \hat{y})H((a_c \hat{t} - M \hat{x}) / (a_c \beta))$$

Therefore

$$\hat{w}(\hat{x}, \hat{y}, t) = g(\beta \hat{x}, \hat{y}), \quad a_c \hat{t} > M \hat{x}$$

$$\hat{w}(\hat{x}, \hat{y}, t) = 0, \quad a_c \hat{t} < M \hat{x}$$

The pressure difference between the upper and lower side of the wing is given by

$$\Delta p = -\rho(Uh_x + h_t)$$

One has, from Equations (4.4) and (4.5)

$$Uh_x + h_t = \left(\frac{U}{\beta}\right) \hat{h}_{\hat{x}} + \frac{M^2}{\beta} \hat{h}_{\hat{t}} + \beta \hat{h}_{\hat{t}}$$

Therefore

$$\Delta p = -\frac{\rho}{\beta}(U\hat{h}_{\hat{x}} + \hat{h}_{\hat{t}})$$

In the wake

$$U\hat{h}_{\hat{x}} + \hat{h}_{\hat{t}} = 0$$

The Lorentz transformation maintains the general structure of the integral equation, but the expressions are somewhat simpler. It requires a transformation of the given upwash.

For the indicial problem, the value of  $\hat{t}$  for which  $\hat{u}$  jumps from zero to the value  $g(\beta\hat{x}, \hat{y})$  now depends upon  $\hat{x}$ . The pressures in the  $\hat{x}, \hat{y}, \hat{t}$  system are expressed by the above formula. After the pressures in the  $\hat{x}, \hat{y}, \hat{t}$  system have been found one must transform back to the  $x, y, t$  system. This is impossible if one tries to apply the results in an aeroelastic problem, for there the displacements of the wing are expressed in terms of the time variable  $t$  while the data obtained by means of the Lorentz transforms appear in terms of  $\hat{t}$ . This is of minor importance, if one first evaluates the indicial response. If one uses the Lorentz transform then the propagation of disturbances occur with the same speed in all directions. This facilitates the choice of the panels on the wing and the timestep, especially at high subsonic Mach numbers. However the speed of propagation may be of importance only during the initial phase of the indicial response. The remainder of the report presents some detailed discussion.

## Section V

### Identity of Equations (3.21) and Equation (VI.18) of Reference 1

The reader is reminded of the following identity. Let  $\xi = \xi(l)$ ,  $\eta = \eta(l)$  describe a curve C, in the  $\xi\eta$  plane and let  $f(\xi, \eta)$  be a given one-valued or vector-valued function. Then

$$d\vec{\xi} \cdot \nabla_2 f = \left( \frac{\partial \xi}{\partial l} \frac{\partial f}{\partial \xi} + \frac{\partial \eta}{\partial l} \frac{\partial f}{\partial \eta} \right) dl = \frac{df}{dl} dl$$

In other words, the operator  $d\vec{\xi} \cdot \nabla$  represents differentiation along the curve C. Hence

$$\int_A^B d\vec{\xi} \cdot \nabla_2 f = \int_A^B \frac{df}{dl} dl = f_B - f_A$$

If the curve is closed, and  $f$  is continuous, then the integral is zero. This holds for a scalar as well as a vector valued function:

$$\oint d\vec{\xi} \cdot \nabla_2 f = \oint \left( \frac{\partial f}{\partial \xi} d\xi + \frac{\partial f}{\partial \eta} d\eta \right) = 0 \quad (5.1)$$

We address ourselves to the first term on the right of Equation (3.20) after Equation (3.21) has been substituted. Let

$$I \triangleq - \frac{\vec{\gamma}(x, y, t)}{4\pi} \cdot \oint \frac{\vec{n}}{R_c} ds \quad (5.2)$$

and

$$\hat{I} \triangleq \lim_{z \rightarrow 0} \frac{dB_8}{dz}$$

where  $\frac{dB_8}{dz}$  is the expression Eq. (VI.18) from Reference 1. We want to show the identity of  $I$  and  $\hat{I}$ . In the present notation,  $\hat{I}$  can be written in the form

$$\hat{I} = \frac{1}{4\pi} \vec{\gamma} \cdot \oint \frac{(\vec{k} \times \vec{\rho})(\vec{\rho} \cdot d\vec{\xi})}{R_c^2 R_c} = \frac{1}{4\pi} \vec{\gamma} \cdot \vec{k} \times \vec{I} \quad (5.3)$$

where

$$\vec{I} = \oint \frac{\vec{\rho}(\vec{\rho} \cdot d\vec{\xi})}{R_c^2 R_c}$$

$\vec{I}$  is rewritten using Equation (3.13)

$$\vec{I} = - \oint \vec{\rho} \left[ d\vec{\xi} \cdot \nabla_2 \left( \frac{1}{R} \right) \right] \quad (5.4)$$

To transform this expression, consider the following integral which is zero because of Equation (5.1).

$$\hat{I} = \oint d\vec{\xi} \cdot \nabla_2 \left( \frac{\vec{\rho}}{R} \right) = 0$$

Applying the scalar operator  $d\xi \cdot \nabla_2 = d\xi \frac{\partial}{\partial \xi} + d\eta \frac{\partial}{\partial \eta}$  to the vector  $\frac{\vec{\rho}}{R_c}$  one obtains

$$\dot{I} = \oint \vec{\rho} \left[ \frac{\partial}{\partial \xi} \left( \frac{1}{R_c} \right) d\xi + \frac{\partial}{\partial \eta} \left( \frac{1}{R_c} \right) d\eta \right] + \oint \frac{1}{R_c} \left( \frac{\partial \vec{\rho}}{\partial \xi} d\xi + \frac{\partial \vec{\rho}}{\partial \eta} d\eta \right) = 0$$

The vector  $\vec{\rho}$  is found in Equation(3.7). One obtains this by carrying out the differentiation in the second term and rewriting the first one.

$$\oint \vec{\rho} d\vec{\xi} \cdot \nabla_2 \left( \frac{1}{R_c} \right) + \oint \frac{1}{R_c} \left[ \left( i + iM \frac{\partial R_c}{\partial \xi} \right) d\xi + \left( iM \frac{\partial R_c}{\partial \eta} + j \right) d\eta \right] = 0$$

According to Equation (5.4), the first term is  $-\vec{I}$ . Therefore

$$\vec{I} = \oint \frac{d\vec{\xi}}{R_c} + iM \oint \left[ \frac{\partial}{\partial \xi} (\log R_c) d\xi + \frac{\partial}{\partial \eta} (\log R_c) d\eta \right]$$

The second term vanishes because the integration is carried out around a closed curve. Therefore

$$\vec{I} = \oint \frac{d\vec{\xi}}{R_c}$$

Now

$$\vec{k} \times d\vec{\xi} = -\vec{n} ds$$

Thus,

$$\vec{k} \times \vec{I} = - \oint \frac{\vec{n} ds}{R_c}$$

This then shows the identity of the expressions  $I$  (Equation (5.1)) and  $\vec{I}$  (Equation (5.3)).

## Section VI

### Steady State Equations

The present formulation can be specialized to the steady state. The equations so obtained are different from those obtained by a direct treatment of the steady state. In this section we show by the procedure of Professor Williams that the two formulations are identical.

In the steady state  $w$  and  $h$  are independent of  $t$ . One obtains from Equations (3.17), (3.18) and (3.19)

$$w(x, y) = w_v(x, y) + w_p \quad (6.1)$$

where

$$w_p = \frac{-U}{2a_c} \frac{\partial h}{\partial x} = \frac{-M}{2} \frac{\partial h}{\partial x}$$

and

$$w_v = \lim_{z \rightarrow 0} \frac{1}{4\pi} \iint \vec{F} \cdot \vec{\gamma} d\xi d\eta$$

Here

$$\vec{F} = \frac{\vec{\rho}}{R_c^2 R_c} = i f_1 + j f_2$$

Substituting  $\vec{\rho}$  from equation (3.8) one obtains

$$f_1 = \frac{1}{\beta^2} \frac{(\xi - x + M R_c)}{R_c^2 R_c} \quad (6.2)$$

$$f_2 = \frac{(\eta - y)}{R_c^2 R_c} \quad (6.3)$$

$$\vec{\gamma} \triangleq i h_\xi + j h_\eta$$

Written in detail Equation (6.1) reads

$$w(x, y) = \frac{1}{4\pi} \lim_{z \rightarrow 0} \iint [f_1(\xi, \eta) h_\xi + f_2(\xi, \eta) h_\eta] d\xi d\eta - \frac{M}{2} h_x(x, y) \quad (6.4)$$

Treating the steady case independently one obtains

$$w(x, y) = \frac{\beta^2}{4\pi} \lim_{z \rightarrow 0} \iint \frac{(\xi - x) h_\xi + (\eta - y) h_\eta}{R_c^3} d\xi d\eta \quad (6.5)$$

It is our aim to establish that the last two equations are in essence identical. Because of the singularity in the denominator, the discussion is carried out for  $z \neq 0$ . Only at the end the limiting process  $z \rightarrow 0$  will be made. The only procedure to transform one equation

into the other is to add some expression to the integrand in Equation (6.4) which can be transformed, by an integration by parts, into an integral around the contour of the region of integration. (This contour integral will not always vanish, but at least the area integrals in Equations (6.4) and (6.5) can be made to agree. The important fact here is that  $f_1$  is the factor of  $h_\xi$  and  $f_2$  the factor of  $h_\eta$ .) From the point of view of operator theory, one makes use of the relation that

$$\int \int \vec{F} \cdot \text{grad } h \, d\xi d\eta = \text{contour terms} - \int \int h \, \text{div} \vec{F} \, d\xi d\eta$$

i.e. that the negative divergence is the adjoint operator to the gradient. (This is readily shown by the integration by parts.) The double integral remains unchanged, if one adds a vector, whose divergence vanishes, to  $\vec{F}$ . In the two-dimensional case such a vector can be written  $i \frac{\partial v}{\partial y} - j \frac{\partial v}{\partial x}$ , where  $v$  is the same scalar function. (In the three-dimensional case, it would be  $\text{curl } \vec{A}$ , where  $\vec{A}$  is some vector valued function.) Accordingly we write, with function  $v$  (so far arbitrary)

$$w_v = \frac{1}{4\pi} \int \int [(f_1 - v_\eta)h_\xi + (f_2 + v_\xi)h_\eta] \, d\xi d\eta + \frac{1}{4\pi} \int \int (v_\eta h_\xi - v_\xi h_\eta) \, d\xi d\eta \quad (6.6)$$

The second integral can be transformed by integrations by part into a contour integral

$$\int \int (v_\eta h_\xi - v_\xi h_\eta) \, d\xi d\eta = \oint h(\xi, \eta) (v_\eta d\eta + v_\xi d\xi) - \int \int h(\xi, \eta) (v_{\eta\xi} - v_{\xi\eta}) \, d\xi d\eta$$

The double integral on the right vanishes. If  $h(\xi, \eta)$  vanishes at the contour of the region under consideration, then the contour integral vanishes, too (otherwise one must carry this expression along).

From the postulate that the double integrals in Equations (6.5) and (6.6) agree, one obtains as one condition

$$f_2(\xi, \eta) + v_\xi = \beta^2 \left( \frac{\eta - y}{R_c^3} \right)$$

Hence, after substitution of  $f_2$  from Equation (6.3)

$$\begin{aligned} v_\xi &= (\eta - y) \left[ \frac{\beta^2}{R_c^3} - \frac{1}{R_c R^2} \right] \\ v_\xi &= \frac{\eta - y}{R_c R^2} [\beta^2 R^2 - R_c^2] \end{aligned} \quad (6.7)$$

and after partial substitution of

$$R = \beta^{-2} [M(\xi - x) + R_c] \quad (6.8)$$

we obtain

$$\psi_\xi = \frac{M(\eta - y)}{\beta^2 R_c^3 R_c^2} [M(\xi - x)^2 + 2R_c(\xi - x) + MR_c^2] \quad (6.9)$$

One remembers that

$$R_c = [(\xi - x)^2 + \beta^2(\eta - y)^2 + \beta^2 z^2]^{\frac{1}{2}} \quad (6.10)$$

From this expression  $\psi$  can be obtained by an integration; the integral encountered in this process exists in a closed form. The simple result has been given by Marc H. Williams.

$$\psi = \frac{-M(\eta - y)}{R_c R_c} \quad (6.11)$$

This can be verified by differentiation. One has

$$\begin{aligned} R_o R_c &= \beta^{-2} [M(\xi - x)R_c + R_c^2] \\ \frac{\partial(R_o R_c)}{\partial \xi} &= \beta^{-2} \left[ MR_c + \frac{M(\xi - x)^2}{R_c} + 2(\xi - x) \right] \\ \frac{\partial(R_o R_c)}{\partial \xi} &= \frac{1}{\beta^2 R_c} [MR_c^2 + 2R_c(\xi - x) + M(\xi - x)^2] \\ \frac{\partial(\frac{1}{R_o R_c})}{\partial \xi} &= \frac{-1}{\beta^2 R_o^2 R_c^2} [M(\xi - x)^2 + 2R_c(\xi - x) + MR_c^2] \end{aligned} \quad (6.12)$$

This vindicates the above choice of  $\psi$ .

One still has to examine the remaining terms in Equation(6.6) (and then those in Equation(6.4)). One has

$$\frac{\partial(R_o R_c)}{\partial \eta} = \frac{(\eta - y)}{R_c} [M(\xi - x) + 2R_c]$$

Then

$$\psi_\eta = -\frac{M}{R_o R_c} + \frac{M(\eta - y)^2}{(R_o R_c)^2 R_c} [M(\xi - x) + 2R_c] \quad (6.13)$$

and using Equation (6.2)

$$f_1 - \psi_\eta = \frac{\xi - x + MR_c}{\beta^2 R_c R_o^2} + \frac{M}{R_o R_c} - \frac{M(\eta - y)^2}{R_o^2 R_c^3} [M(\xi - x) + 2R_c]$$

$$f_1 - \psi_\eta = \frac{1}{\beta^2 R_c^3 R_o^2} [(\xi - x)R_c^2 + MR_c^3 + M\beta^2 R_o R_c^2 - M\beta^2(\eta - y)^2 [M(\xi - x) + 2R_c]]$$

Here

$$\beta^2 R_o = M(\xi - x) + R_c$$



and

$$\beta^2(\eta - y)^2 = R_c^2 - (\xi - x)^2 - \beta^2 z^2$$

are substituted. Then the expression reduces to

$$f_1 - \psi_\eta = \frac{1}{R_c^3} \frac{\xi - x}{\beta^2 R_c^2} \left[ M^2 (\xi - x)^2 + 2MR_c(\xi - x) + R_c^2 \right] + \frac{Mz^2 [M(\xi - x) + 2R_c]}{R_c^3 R_o^2}$$

The term within the first bracket on the right is  $\beta^4 R_c^2$ . Hence

$$f_1 - \psi_\eta = \frac{\beta^2(\xi - x)}{R_c^3} + \frac{Mz^2 [M(\xi - x) + 2R_c]}{R_c^3 R_o^2}$$

At least the first term has the expected form. So far we have found

$$\begin{aligned} w(x, y) = & \frac{\beta^2}{4\pi} \iint \frac{(\xi - x)h_\xi + (\eta - y)h_\eta}{R_c^3} d\xi d\eta + \oint h(\xi, \eta)(\psi_\eta d\eta + \psi_\xi d\xi) \\ & + \lim_{z \rightarrow 0} \frac{1}{4\pi} Mz^2 \iint \frac{\partial h}{\partial \xi} \frac{1}{R_o^2 R_c^2} \left[ 2 + \frac{M(\xi - x)}{R_c} \right] d\xi d\eta - \frac{M}{2} \frac{\partial h}{\partial x} \end{aligned} \quad (6.14)$$

here  $\psi_\xi$  and  $\psi_\eta$  are found in Equations(6.7), (6.8), and (6.11).

The first term on the right is the one familiar from steady state theory. It will be shown that the last two terms cancel in the limit  $z \rightarrow 0$ . The results are summarized in the form

$$\begin{aligned} & \frac{1}{4\pi} \iint_A (f_1 h_\xi + f_2 h_\eta) d\xi d\eta - \frac{M}{2} h_x(x, y) = \\ & = \frac{\beta^2}{4\pi} \iint_A \frac{(\xi - x)h_\xi + (\eta - y)h_\eta}{R_c^3} d\xi d\eta + \frac{1}{4\pi} \oint_{\Gamma_A} h(\xi, \eta)(\psi_\xi d\xi + \psi_\eta d\eta) \end{aligned} \quad (6.15)$$

where  $f_1, f_2, \psi, \psi_\xi, \psi_\eta, R_o$ , and  $R_c$  are found in Equations (6.2), (6.3), (6.9), (6.7), (6.11), (6.10), and (6.9).

To show the last two terms in Equation (6.14) cancel in the limit  $z \rightarrow 0$ , one must consider that because of the factor  $z^2$ , the contribution of the second to last term in Equation (6.14) vanishes everywhere except for the immediate vicinity of the point  $(x, y)$ . Thus  $\frac{\partial h}{\partial \xi}$  can be replaced by  $\frac{\partial h}{\partial x}$  and the second to last term in Equation (6.14) simplifies to

$$\frac{\partial h}{\partial x} \cdot \frac{M}{4\pi} I \quad (6.16)$$

where

$$I = \iint \frac{z^2}{R_o^2 R_c^3} [2R_c + M(\xi - x)] d\xi d\eta \quad (6.17)$$

The integration is extended over the interior of a small ellipse given by

$$(\xi - x)^2 + \beta^2(\eta - y)^2 = \beta^2 \rho_c^2 \quad \rho_c = \text{constant} \quad (6.18)$$

where  $\rho \neq 0$  is small. Outside of the ellipse, the limit as  $z \rightarrow 0$  is 0. According to Equations (3.5) and (3.6)

$$R = \frac{1}{\beta^2} (R_c + M(\xi - x))$$

$$R_c = ((\xi - x)^2 + \beta^2(\eta - y)^2 + \beta^2 z^2)^{\frac{1}{2}}$$

and let

$$\xi - x = \beta z \hat{\xi}$$

$$\eta - y = z \hat{\eta}$$

$$R_c = z\beta(\hat{\xi}^2 + \hat{\eta}^2 + 1)^{\frac{1}{2}}$$

$$R_c = \frac{z}{\beta} \left[ (\hat{\xi}^2 + \hat{\eta}^2 + 1)^{\frac{1}{2}} + M\hat{\xi} \right]$$

$$d\xi d\eta = z^2 \beta d\hat{\xi} d\hat{\eta}$$

Then the boundary of the ellipse, given by Equation (6.18) becomes  $\hat{\xi}^2 + \hat{\eta}^2 = \frac{\rho_c^2}{z^2}$ . In the limit  $z \rightarrow 0$  the radius for this circle tends to infinity in the  $\hat{\xi}, \hat{\eta}$ -plane. The values of  $z$  in the integral cancel and one obtains

$$I = \int \int \frac{\beta(2\sqrt{\hat{\xi}^2 + \hat{\eta}^2 + 1} + M\hat{\xi})}{\frac{1}{\beta^2}(\sqrt{\hat{\xi}^2 + \hat{\eta}^2 + 1} + M\hat{\xi})^2 \beta^3(\hat{\xi}^2 + \hat{\eta}^2 + 1)^{\frac{3}{2}}} \beta d\hat{\xi} d\hat{\eta}$$

Now let

$$\hat{\xi} = \rho \cos \theta$$

$$\hat{\eta} = \rho \sin \theta$$

Then

$$d\hat{\xi} d\hat{\eta} = \rho d\rho d\theta$$

and one obtains

$$I = \beta \int_0^\infty \frac{\rho}{(\rho^2 + 1)^{\frac{3}{2}}} I_1(\rho) d\rho \quad (6.19)$$

where

$$I_1(\rho) = \int_{-\pi}^{\pi} \frac{2\sqrt{\rho^2 + 1} + M\rho \cos \theta}{(\sqrt{\rho^2 + 1} + M\rho \cos \theta)^2} d\theta \quad (6.20)$$

The following formula can be found from the integral tables, for instance Reference 3.

$$\int \frac{A + B \cos x}{(a + b \cos x)^2} dx = \frac{1}{a^2 - b^2} \left[ (aB - bA) \frac{\sin x}{a + b \cos x} + (aA - bB) \int \frac{dx}{a + b \cos x} \right]$$

It can, of course, be verified by differentiating the right hand side. The variable  $x$  corresponds to  $\theta$ . For the limits  $-\pi$  and  $+\pi$  the first term on the right vanishes and one obtains

$$\int_{-\pi}^{+\pi} \frac{A + B \cos x}{(a + b \cos x)^2} dx = \frac{aA - bB}{a^2 - b^2} \int_{-\pi}^{+\pi} \frac{1}{a + b \cos x} dx$$

Furthermore one has

$$\int_{-\pi}^{+\pi} \frac{1}{a + b \cos x} dx = \frac{2}{\sqrt{a^2 - b^2}} \operatorname{arctg} \frac{\sqrt{a^2 - b^2} (\operatorname{tg} \frac{x}{2})}{a + b}$$

(again verified by differentiating). Therefore

$$\int_{-\pi}^{+\pi} \frac{1}{a + b \cos x} dx = \frac{2\pi}{\sqrt{a^2 - b^2}}$$

Hence

$$\int_{-\pi}^{+\pi} \frac{A + B \cos \theta}{(a + b \cos \theta)^2} d\theta = 2\pi \frac{aA - bB}{(a^2 - b^2)^{\frac{3}{2}}}$$

Applying this formula to Equation (6.20), one has

$$A = 2\sqrt{\rho^2 + 1}$$

$$B = M\rho$$

$$a = \sqrt{\rho^2 + 1}$$

$$b = M\rho$$

Hence

$$aA - bB = 2(\rho^2 + 1) - M^2\rho^2 = (2 - M^2)\rho^2 + 2$$

$$a^2 - b^2 = \rho^2 + 1 - M^2\rho^2 = \beta^2\rho^2 + 1$$

Thus

$$I_1(\rho) = 2\pi \frac{(2 - M^2)\rho^2 + 2}{(\beta^2\rho^2 + 1)^{\frac{3}{2}}} \quad (6.21)$$

and, from Equation (6.19)

$$I = 2\pi\beta \int_0^\infty \frac{((2 - M^2)\rho^2 + 2)\rho^{\frac{3}{2}}}{(\rho^2 + 1)^{\frac{3}{2}}(\beta^2\rho^2 + 1)} d\rho$$

Let

$$\rho^2 = u$$

$$2\rho d\rho = du$$

Then

$$I = \pi\beta \int_0^\infty \frac{(2 - M^2)u + 2}{[\beta^2 u^2 + (\beta^2 + 1)u + 1]^{\frac{3}{2}}} du \quad (6.22)$$

which one can find in the tables of integrals. For  $X = ax^2 + 2bx + c$

$$\int \frac{x}{X^{\frac{3}{2}}} dx = -\frac{1}{ac - b^2} \frac{bx + c}{X^{\frac{1}{2}}}$$

$$\int \frac{1}{X^{\frac{3}{2}}} dx = -\frac{1}{ac - b^2} \frac{ax + b}{X^{\frac{1}{2}}}$$

In the present case

$$a = \beta^2$$

$$b = \frac{\beta^2 + 1}{2} = 1 - \frac{M^2}{2}$$

$$c = 1$$

$$ac - b^2 = -\frac{M^4}{4}$$

$$\int_0^\infty \frac{x}{X^{\frac{3}{2}}} dx = \frac{4}{M^4} \left[ \frac{(\frac{\beta^2+1}{2})x + 1}{(\beta^2 x^2 + (\beta^2 + 1)x + 1)^{\frac{1}{2}}} \right] \Big|_0^\infty$$

$$\int_0^\infty \frac{x}{X^{\frac{3}{2}}} dx = \frac{4}{M^4} \left[ \frac{1 - (\frac{M^2}{2})}{\beta} - 1 \right]$$

$$\int_0^\infty \frac{1}{X^{\frac{3}{2}}} dx = -\frac{4}{M^4} \left[ \frac{\beta^2 x + \frac{\beta^2+1}{2}}{(\beta^2 x^2 + (\beta^2 + 1)x + 1)^{\frac{1}{2}}} \right] \Big|_0^\infty$$

$$\int_0^\infty \frac{1}{X^{\frac{3}{2}}} dx = \frac{4}{M^4} (1 - \frac{M^2}{2} - \beta)$$

With these formulae one obtains from Equation (6.22)

$$I = \frac{4\pi\beta}{M^4} \frac{(2 - 2M^2 + \frac{M^4}{2} - 2 + 2M^2)}{\beta}$$

$$I = 2\pi$$

To complete the proof that the last two terms in Equation (6.14) cancel, we rewrite Equation (6.16) (the second to last term in Equation (6.14)).

$$\frac{\partial h}{\partial x} \frac{M}{4\pi} I$$

and substitute our result for  $I$

$$\frac{M}{2} h_r$$

which cancels with the last term in Equation (6.14).

## Section VII

### References

1. Guderley, Karl G. and Blair, Maxwell. "The Integral Equation for the Time-Dependent Linearized Potential Flow Over a Wing." AFWAL-TR-86-3077. Flight Dynamics Laboratory, Air Force Wright Aeronautical Laboratories, Air Force Systems Command, December 1986.
2. Lomax, Harvard. Heaslet, Max. A., Fuller, Franklyn B., Sluder, Loma. "Two- and Three-Dimensional Unsteady Lift Problems in High-Speed Flight." NACA Report 1077. 1952.
3. Grobner, W., Hofreider, M., Hofreider, N., Laub, J., Laub, Peschel, E., "Integraltafel. Unbestiminide Integrale." Zentrale fur Wissenschaftliches Berichtswesen der Luftfahrtforschung, Berlin-Aldershof. 1944.
4. Blair, M., Williams, M.H., "A Time Domain Panel Method for Wings." Proceedings of the AIAA/ASME/ASCE/AHS 30th Structures, Structural Dynamics, and Materials Conference, AIAA paper #89-1323, April 1989.

## Appendix

### Identity of the Present Equations (3.17), (3.18), and (3.22) with Reference 1

In reference 1 dimensionless quantities are used while in the present derivation the quantities in their original form are used. Equations (3.17), (3.18), and (3.22) correspond to Equation (VI.22) in reference 1. For the present discussion, we characterize the quantities of reference 1 by a tilde.

They are

$$\tilde{u}(\tilde{t}, \tilde{x}, \tilde{y}) = \tilde{u}_p(\tilde{t}, \tilde{x}, \tilde{y}) + \tilde{u}_r(\tilde{t}, \tilde{x}, \tilde{y})$$

$$\tilde{u}_p(\tilde{t}, \tilde{x}, \tilde{y}) = \frac{-2\pi}{(1 - M^2)^{\frac{1}{2}}} \left[ \frac{\partial \tilde{h}}{\partial \tilde{x}}(\tilde{t}, \tilde{x}, \tilde{y}) + M \frac{\partial \tilde{h}}{\partial \tilde{t}}(\tilde{t}, \tilde{x}, \tilde{y}) \right] \quad (A.1)$$

$$\tilde{u}_r = \int \int \frac{(\tilde{\xi} - \tilde{x} + M\tilde{\rho})\tilde{h}^{(2)}(\tilde{\tau}, \tilde{\xi}, \tilde{\eta}) + (1 - M^2)(\tilde{\eta} - \tilde{y})\tilde{h}^{(3)}(\tilde{\tau}, \tilde{\xi}, \tilde{\eta})}{\tilde{\rho} [M(\tilde{\xi} - \tilde{x}) + \tilde{\rho}]^2} d\tilde{\xi} d\tilde{\eta} \quad (A.2)$$

Here  $\tilde{\rho} = [(\tilde{\xi} - \tilde{x})^2 + (\tilde{\eta} - \tilde{y})^2]^{\frac{1}{2}}$

$$\tilde{u} = \frac{\partial \tilde{\phi}}{\partial \tilde{z}}, \quad \tilde{\tau} = \tilde{t} - \frac{1}{1 - M^2} [M(\tilde{\xi} - \tilde{x}) + \tilde{\rho}]$$

and at the wing

$$\tilde{\phi}(\tilde{t}, \tilde{x}, \tilde{y}, 0) = 2\pi \tilde{h}(\tilde{t}, \tilde{x}, \tilde{y})$$

The notation  $\tilde{h}^{(2)}$  and  $\tilde{h}^{(3)}$  refers to the derivatives of  $\tilde{h}(\tilde{\tau}, \tilde{\xi}, \tilde{\eta})$  with respect to its second and third argument respectively. We now turn to the present formulation. To make the analogy between the present formulation and that of reference 1 more evident, we write

$$R_c \Big|_{z=0} = \rho$$

$$\rho = [(\xi - x)^2 + (1 - M^2)(\eta - y)^2]^{\frac{1}{2}}$$

Then

$$R_c = \frac{1}{1 - M^2} [M(\xi - x) + \rho]$$

The vector  $\vec{\rho}$  (Equation (3.18)) then appears as

$$\frac{i}{\rho^2} [\xi - x + M\rho] + j(\eta - y)$$

Furthermore, we write  $h(\tau, \xi, \eta)$  instead of  $h(\xi, \eta, \tau)$  and denote by  $h^{(2)}$  and  $h^{(3)}$  the derivatives of  $h$  with respect to the second and third argument. Then

$$\nabla_2 h = i h^{(2)} + j h^{(3)}$$

Then Equations (3.17), (3.218), and (3.19) assume the form

$$u' = u'_v + u'_p$$

$$u'_p = -\frac{1}{2\pi a_\infty} (h_t(t, x, y) + U h_x(t, x, y)) \quad (A.3)$$

$$u'_v = \frac{1}{4\pi} (1-M^2) \int \int \frac{(\xi - x + M\rho) h^{(2)}(\tau, \xi, \eta) + (1-M^2)(\eta - y) h^{(3)}(\tau, \xi, \eta)}{\rho [M(\xi - x) + \rho]^2} d\xi d\eta \quad (A.4)$$

$$\tau = t - \frac{1}{a_\infty(1-M^2)} [M(\xi - x) + \rho]$$

$$\rho = [(\xi - x)^2 + (1-M^2)(\eta - y)^2]^{\frac{1}{2}}$$

$$w = \frac{\partial \phi}{\partial z}$$

On the upper side of the wing

$$\phi = (t, x, y) = \frac{1}{2} h(t, x, y)$$

(see discussion preceeding, Equation (2.12)).

The difference between the two formulations is brought about by the fact that in reference 1 the authors deal with dimensionless quantities, and apply the Prandtl-Glauert coordinate transformation. (In unsteady problems the Prandtl-Glauert transformation is less useful and in a later report Guderley abandoned it.) Moreover the definitions of  $h$  differ by a factor of  $4\pi$ . The quantities in Reference 1 are made dimensionless with a characteristic length,  $L$ , the freestream velocity,  $U$ , and the freestream velocity of sound,  $a$ .

One has

$$\begin{aligned} x &= \tilde{x}L & \xi &= \tilde{\xi}L \\ y &= \frac{\tilde{y}L}{\beta} & \eta &= \frac{\tilde{\eta}L}{\beta} \end{aligned}$$

$$z = \frac{\tilde{z}L}{\beta}$$

$$t = \frac{\tilde{t}L}{a_\infty}$$

$$\phi(t, x, y, z) = UL\tilde{\phi}(\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z})$$

$$h(t, x, y) = 4\pi UL\tilde{h}(\tilde{t}, \tilde{x}, \tilde{y})$$



$$h(t, x, y) = 4\pi U L \hat{h} \left( \frac{ta}{L}, \frac{x}{L}, \frac{y\beta}{L} \right)$$

Therefore

$$\frac{\partial \phi}{\partial z} = U\beta \frac{\partial \hat{\phi}}{\partial \hat{z}} \quad w = U\beta \hat{w}$$

$$d\xi = L d\hat{\xi} \quad w_p = U\beta \hat{w}_p$$

$$d\eta = \frac{L d\hat{\eta}}{\beta} \quad w_v = U\beta \hat{w}_v$$

$$\rho = L \left[ (\dot{\hat{\xi}} - \dot{\hat{x}})^2 + (\dot{\hat{\eta}} - \dot{\hat{y}})^2 \right] = L \hat{\rho}$$

$$\frac{L}{a} \left[ \dot{\hat{t}} - \frac{1}{1-M^2} \left( M (\dot{\hat{\xi}} - \dot{\hat{x}}) \right) + \dot{\hat{\rho}} \right] = \frac{L}{a} \hat{\tau}, \quad \hat{\tau} = \frac{\tau a}{L}$$

$$h(\tau, \xi, \eta) = 4\pi U L \hat{h}(\hat{\tau}, \hat{\xi}, \hat{\eta})$$

and

$$h^{(2)}(\tau, \xi, \eta) = 4\pi U L \hat{h}^{(2)}$$

$$h^{(3)}(\tau, \xi, \eta) = 4\pi \beta U L \hat{h}^{(3)}$$

$$(\eta - y) h^{(3)}(\tau, \xi, \eta) = 4\pi U (\hat{\eta} - \hat{y}) L \hat{h}^{(3)} L$$

$$(\xi - x + M\rho) = L(\hat{\xi} - \hat{x} + M\hat{\rho})$$

$$[M(\xi - x) + \rho] = L [M(\hat{\xi} - \hat{x}) + \hat{\rho}]$$

Since

$$h_x = h^{(2)} = 4\pi U \hat{h}_{\hat{x}}$$

$$h_t = h^{(3)} = 4\pi U a \hat{h}_{\hat{t}}$$

$$U\beta \hat{w}_p = -2\pi U (\hat{h}_{\hat{t}} + M\hat{h}_{\hat{x}})$$

Substituting these quantities into Equations (A.3) and (A.4) one indeed obtains Equations (A.1) and (A.2).